REFLECTION OF A WAVE FROM A BOUNDARY COMPOSED OF ARCS OF VARIABLE CURVATURE

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A plane problem of reflection of an acoustic wave from a curvilinear, variable-curvature boundary is considered. The curvature of the boundary or its derivative of some order has a discontinuity at one point. This gives rise to an additional, diffracted wave, with a circular wavefront centered at this point. The first term of the geometric acoustical expansion of this wave is determined and a strict estimate of the order of magnitude of the neglected terms is given.

1. Let D denote a plane region $y > g(x)$ with the boundary Γ $(y = g(x))$, on which

$$g(x) = g_0(x) \quad (x \leqslant 0), \qquad g(x) = g_0(x) + \psi(x) \quad (x \geqslant 0) \tag{1.1}$$

The functions g_0 and ψ are analytic and near the point $x = 0$ we have

$$\psi(x) = ax^n / n! + O(x^{n+1}),$$
$$\psi'(x) = ax^{n-1} / (n-1)! + O(x^n), \quad n > 1 \tag{1.2}$$

where n is an integer. Thus $g^{(n)}(x)$ has a discontinuity equal to a at $x = 0$. Let a wave for which the first term of the geometrical acoustical expansion

$$u(t, x, y) = A(x, y) f_0(t - p(x, y)) + \ldots \tag{1.3}$$

is known, be propagated in the region D.

A solution is sought to the problem of the reflection of this wave from the boundary Γ, i.e. the determination of a function U satisfying the wave equation $U_{tt} = U_{xx} + U_{yy}$, in D and one of the boundary conditions

$$U = 0 \quad \text{or} \quad \partial U / \partial n = 0 \tag{1.4}$$

on Γ, and identical with the incident wave (1.3) in the region not yet reached by the reflected wave (the reflected wavefront is obtained by the usual methods).

The required function U can be written as a sum $U = u + v + w$, where u is the incident wave (1.3), v is the wave formed on reflection of u from the boundary Γ_0 $(y = g_0(x))$. The wave v can be obtained by the usual ray-tracing method. The function w is a correction arising from the fact that the sum $u + v$ satisfies the boundary condition on Γ_0 and not on Γ. In order that the function U should satisfy the above requirements, the following conditions must be met. Function w must satisfy the wave equation of one of the boundary conditions

$$w = -q(t, x), \quad q = (u + v)|_\Gamma \quad \text{or} \quad \frac{\partial w}{\partial n} = -q(t, x), \quad q = \left.\frac{\partial(u + v)}{\partial n}\right|_\Gamma$$

on Γ and must become $w = 0$ in the region not yet reached by the waves reflected from Γ and Γ_0. Since Γ coincides with Γ_0 for $x \leqslant 0$, we have $q(t, x) = 0$.

The solution of the wave equation with boundary condition $\partial w / \partial n = -q(t, x)$ is expressed by the Green's function G as

$$w(T, X, Y) = \iint G(T - t, X, Y; x, y)\, q(t, x)\, dt ds \tag{1.5}$$

where the integration is performed over that part of Γ where $Gq \neq 0$, and ds is the differential of the arc length of Γ.

The Green's function G is represented as a sum of the Green's function G_0 for the region bounded by Γ_0 together with the correction $G_1 = G - G_0$. The function $G_0(t_1, X, Y; x, y)$ regarded here as a function of t_1, x, y, is the sum of the circular wave G_2 originating at a point source (X, Y) and the wave G_3 formed on reflection of G_2 from Γ_0. The wave G_2 is known, the geometric acoustical expansion of the wave G_3 is obtained by the familiar method and the correction G_1, as shown in Sects. 7 and 8, does not affect the first term of the geometric acoustical expansion of the wave w.

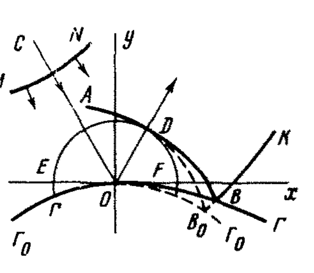

Fig. 1

2. Consider first the problem with the boundary condition $\partial U / \partial n = 0$ and obtain the value of $\partial (u + v) / \partial n$ on Γ (see Fig. 1). Let the x-axis touch the boundary at the point O, the wavefront MN of the incident wave (1.3) reach the point O at the instant $t = 0$ and the ray CO form an angle β with the x-axis ($0 < \beta < \pi$). The direction of curvature of the boundary over each of its segments $x > 0$ and $x < 0$ is not important, as is the relative location of the curves Γ and Γ_0 for $x > 0$. When $t > 0$, BK is the incident wavefront, ADB is the wavefront reflected from Γ, ADB_0 denotes the position of the reflected wavefront in the case when Γ_0 serves as the boundary, and EDF is the circular wavefront of the diffracted wave w_0 (the existence of such a wave was noted in [1], pp. 23-24).

Suppose that in the geometric acoustical expansion of the incident wave (1.3) we have

$$f_0(\tau) = 0 \quad (\tau \leqslant 0) \qquad f_0(\tau) = \tau^m / \Gamma(m+1) \quad (\tau > 0) \tag{2.1}$$

where $m \geqslant 1$, Γ is the gamma function and $\tau = t - p(x, y)$.

The wave reflected from Γ_0 has the following geometric acoustical expansion

$$v = B(x, y) f_0(\tau_0) + \ldots, \qquad \tau_0 = t - p_0(x, y) \tag{2.2}$$

Let n and n_0 be the directions normal to Γ and Γ_0, and ∇p and ∇p_0 the gradients of the functions p and p_0 which are, of course, directed along the rays. Furthermore n, n_0 and ∇p_0 form acute angles with the y-axis, and ∇p an obtuse angle.

Then $\partial \tau / \partial n = -\partial p / \partial n = -\cos(\nabla p, n)$ and a similar relation is true for $\partial \tau / \partial n_0$. At the point $x = y = 0$ we have

$$p = p_0 = 0, \qquad \partial p / \partial y = \cos(\nabla p, y) = -\sin\beta, \qquad \partial p_0 / \partial y = \sin\beta$$

therefore on the curves Γ and Γ_0 near this point as well as between them we have

$$\partial p / \partial y = -\sin\beta + O(x), \qquad \partial p_0 / \partial y = \sin\beta + O(x) \tag{2.3}$$

From (1.3) and (2.2) it follows that on Γ

$$\frac{\partial u}{\partial n} = -A\gamma f_0'(\tau) + \left(P + \frac{\partial Q}{\partial n}\right) f_0(\tau), \quad \gamma = \cos(\nabla p, n) \tag{2.4}$$

$$\frac{\partial v}{\partial n} = -B\gamma_0 f_0'(\tau_0) + \left(R + \frac{\partial S}{\partial n}\right) f_0(\tau_0), \quad \gamma_0 = \cos(\nabla p_0, n)$$

Here P, Q, R, S are functions of t, x, y with bounded derivatives. On Γ_0 the derivatives $\partial u / \partial n_0$ and $\partial v / \partial n_0$ are expressed in a similar way. Since v is the wave reflected from Γ_0, we have on Γ_0

$$\partial (u + v) / \partial n_0 = 0, \ \tau_0 = \tau, \ \gamma_0 = -\gamma, \ B = A \qquad (2.5)$$

Let us determine how the values of these functions change on passing from a point on Γ_0 to a point on Γ possessing the same abscissa. (The functions entering the geometric acoustical expansion are analytically continuable to some vicinity of the curve Γ_0). The distance between these points is equal to $\psi(x)$ and the angle between the normals at these points is equal to $\psi'(x) + O(x\psi')$. It follows, that during the passage between the two boundaries the quantities $A, B, P, Q, R, S, \tau, \tau_0$ change by $O(\psi)$, the normal derivatives by $O(\psi')$, the terms containing f_0 in (2.4) by $O(\psi'\tau^m)$, and the quantities γ and γ_0 by $\psi'(x) \times \cos\beta + O(x^n)$. From these estimates and from (1.2) and (2.5) it follows that on Γ

$$\gamma + \gamma_0 = 2\psi'(x) \cos\beta + O(x^n)$$

$$\frac{\partial (u + v)}{\partial n} = -A\gamma f_0'(\tau) - B\gamma_0 f_0'(\tau_0) + O(d^{n+m-1}) \qquad (d = |t| + |x|)$$

The sum of the terms containing A and B can now be written as

$$B\gamma_0 [f_0'(\tau) - f_0'(\tau_0)] - [A(\gamma + \gamma_0) + \gamma_0 (B - A)] f_0'(\tau)$$

Using the previous estimates we have on Γ for $x \geqslant 0$

$$\frac{\partial (u + v)}{\partial n} = (A_0 \sin\beta + a_0)(f_0'(\tau) - f_0'(\tau_0)) - 2A_0 f_0'(\tau) \psi' \cos\beta + R_0 \qquad (2.6)$$

$$A_0 = A(0, 0), \ a_0 = B\gamma_0 - A_0 \sin\beta = O(x), \ R_0 = O(d^{n+m-1})$$

It should be noted that on Γ, i.e. when $y = g(x)$, we have

$$\tau = t - p_1(x), \tau_0 = t - p_2(x), \qquad p_1(x) = p(x, g(x))$$
$$p_2(x) = p_0(x, g(x)) \qquad (2.7)$$

and similarly to (2.3)

$$p_i'(x) = -\cos\beta + O(x), \ p_i(x) = -x\cos\beta + O(x^2), \ i = 1, 2 \qquad (2.8)$$

Since by (2.5) $\tau_0 = \tau$ on Γ_0, then $p_0 = p$ for $y = g_0(x)$. Equations (2.3) now imply that when $y = g_0(x) + \psi(x)$, i.e. on Γ

$$p_0 - p = p_2(x) - p_1(x) = 2\psi(x) \sin\beta + O(x\psi) \qquad (2.9)$$

3. Let us now find the principal part of the Green's function in (1.5). By Sect. 1 we have

$$G = G_0 + G_1, \ G_0 = G_2 + G_3 \qquad (3.1)$$

$$G_2(t_1, X, Y; x, y) = (2\pi)^{-1} (t_1^2 - \rho^2)^{-1/2}$$

$$\rho = [(x - X)^2 + (y - Y)^2]^{1/2}$$

The wave G_2, regarded as a function of t_1, x, y, has the following geometric acoustical expansion

$$G_2 = A^* (t_1 - \rho)^{-1/2} + O((t_1 - \rho)^{1/2}), \qquad A^* = (2\pi\sqrt{2\rho})^{-1} \qquad (3.2)$$

Here and below, it is implied that expressions such as $a^{-1/2}$, $a^{1/2}$ and similar are replaced by zero when $a \leqslant 0$.

Since G_3 is a wave generated by reflection of the wave (3.2) from Γ_0, we find that, as in Sect. 2,

$$G_3 = B^* \, (t_1 - \rho_0)^{-1/2} + O\,((t_1 - \rho_0)^{1/2}), \qquad B^* = A^* + O\,(\psi)$$
$$\rho_0 = \rho + O\,(\psi) \tag{3.3}$$

on Γ.

At the point (0, 0) we have

$$A^* = B^* = A_0^*, \ \ A_0^* = (2\pi \sqrt{2r})^{-1}, \ \ r^2 = X^2 + Y^2$$

When x is small

$$A^* = A_0^* + O\,(x), \ \ B^* = A_0^* + O\,(x)$$

at any point on Γ.

Setting

$$t_1 = T - t, \qquad T = r + h, \qquad X = r\cos\varphi, \qquad Y = r\sin\varphi \tag{3.4}$$

we obtain

$$\rho = r - x\cos\varphi - y\sin\varphi + O\,(x^2 + y^2)$$

On Γ, i.e. when $y = g\,(x) = O\,(x^2)$

$$t_1 - \rho = p_3\,(x) + h - t, \qquad t_1 - \rho_0 = p_4\,(x) + h - t$$
$$p_j\,(x) = x\cos\varphi + O\,(x^2), \quad p_j'\,(x) = \cos\varphi + O\,(x) \quad (j = 3,4) \tag{3.5}$$

where the derivatives are estimated in a similar fashion to those of (2.3).

The form of the region of integration in (1.5) is discussed next. According to Sect.1, $q = 0$ when $x \leqslant 0$. Furthermore $G = 0$ on Γ, when $t > h + \max\,(p_3, p_4)$, and the waves u and v arrive at the point $(x, g\,(x))$ on Γ at the instants $p_1\,(x)$, $p_2\,(x)$ (see (2.7)). Therefore the integration in (1.5) is performed over the region

$$x > 0, \qquad \min\,(p_1, p_2) < t < h + \max\,(p_3, p_4) \tag{3.6}$$

By (2.8) and (3.5) the boundaries of this region differ from the straight lines $t = -x\cos\beta$ and $t = h + x\cos\varphi$ only by $O\,(x^2)$ when $x > 0$.

Assume that $\varphi > \pi - \beta$ and h is sufficiently small. Then for $h \leqslant 0$ we have that $Gq = 0$ everywhere in (1.5), i.e. $w = 0$ when $T \leqslant r$.

When $h > 0$, the curves $t = p_i\,(x)$ and $t = h + p_j\,(x)$ intersect the abscissa at the point x_{ij}, i.e.

$$p_i\,(x_{ij}) = h + p_j\,(x_{ij}) \qquad (i = 1, 2; \ j = 3, 4) \tag{3.7}$$

Formulas (2.8) and (3.5), together with the inequality $\varphi > \pi - \beta$, now yield

$$x_{ij} = hb^{-1} + O\,(h^2), \qquad b = -\cos\beta - \cos\varphi > 0 \tag{3.8}$$

This means that the quantities t, x, y, d in (3.6) are of the order of $O\,(h)$.

4. The principal part of the diffracted wave can be found by replacing G by G_0 in (1.5), i.e. neglecting G_1. If, in additior. R_0 is neglected together with a_0 in (2.6) as well as the remainders in (3.2) and (3.3), replacing ds by dx and A^*, B^* by A_0^*, gives rise to an error of the order $O\,(h^{n+m+1/2})$. It remains to calculate the integrals (for $j = 3, 4$)

$$L_{j,m} = \iint \frac{f_0'(\tau)\,\psi'(x)}{(p_j + h - t)^{1/2}}\,dt\,dx, \qquad M_{j,m} = \iint \frac{f_0'(\tau) - f_0'(\tau_0)}{(p_j + h - t)^{1/2}}\,dt\,dx$$

The quantities τ, τ_0, f_0 appearing here are the same as in (2.7) and (2.1). The integrals are computed first for $m = 1/2$, then for any $m > 1/2$. Integration with respect to t from $p_1\,(x)$ to $h + p_j\,(x)$ yields

$$\pi^{-1/2} L_{j,1/2} = \int_0^{x_{1j}} \psi'(x)\,dx = \psi\,(x_{1j}) = \psi\,(h/b) + O\,(h^2\psi') \tag{4.1}$$

where x_{ij} are the same as in (3.7) and (3.8). Similarly we obtain

$$\pi^{-1/2} M_{j,1/2} = x_{1j} - x_{2j}$$

Let us estimate this difference. From (3.7) it follows

$$p_j(x_{2j}) - p_j(x_{1j}) + p_2(x_{1j}) - p_2(x_{2j}) = p_2(x_{1j}) - p_1(x_{1j})$$

Applying the finite increment formula to the left-hand side of the above expression, using the estimates for the derivatives (2.8) and (3.5), and applying (2.9) and (3.8) to its right-hand side, we obtain

$$(x_{2j} - x_{1j})(\cos\varphi + \cos\beta + O(x_{1j} + x_{2j})) = 2\psi(x_{1j})\sin\beta +$$
$$+ O(x_{1j}\psi(x_{1j})),\ x_{1j} - x_{2j} = 2b^{-1}\psi(hb^{-1})\sin\beta + O(h\psi) \qquad (4.2)$$

Now consider the case for any $m > 1/2$. The operator

$$I_k f(h) = \int_{-\infty}^{h} \frac{(h-s)^{k-1}}{\Gamma(k)} f(s)\,ds \qquad (k > 0) \qquad (4.3)$$

transforms a function of the form (2.1) into an analogous function in which m is replaced by $m + k$. In particular, application of the operator I_k, where k is an integer, to any function is equivalent to integrating this function k times.

Applying the operator I_k where $k = m - 1/2$ to the function $L_{j,1/2}(h)$ and taking (1.2) into account, we obtain

$$L_{j,m}(h) = \sqrt{\pi}\, ab^{-n} h^{n+m-1/2} / \Gamma(n + m + 1/2) + O(h^{n+m+1/2})$$

The formula for $M_{j,m}$ differs only by the factor $2b^{-1}\sin\beta$.

Using the expressions for the integrals and neglecting G_1 (this will be justified in Sect. 7), we obtain the following expression for the diffracted wave for $T > r$ under the boundary condition $\partial U / \partial n = 0$:

$$w_0(T, X, Y) = \frac{\sqrt{2}\, a A_0 h^{n+m-1/2}(1 + \cos\beta\cos\varphi)}{\sqrt{\pi r}\,\Gamma(n + m + 1/2)(-\cos\beta - \cos\varphi)^{n+1}} + O(h^{n+m+1/2}) \qquad (4.4)$$

Here $h = T - r$; r, φ are polar coordinates of the control point (X, Y); a and n depend on the form of the boundary (see (1.2)); $A_0 = A\ (0, 0)$ and m depend on the incident wave (see (1.3) and (2.1)). Finally β is the polar angle of the direction of the incident ray arriving at the point (0, 0). Thus, when $T \leqslant r$ we have $w_0 = 0$.

The formula holds near the front of the diffracted wave, at some distance from the reflected ray $\varphi = \pi - \beta$ and the boundary Γ. If on the other hand the boundary Γ coincides with the line $\varphi = \pi$, when $x \leqslant 0$, then formula (4.4) holds up to this part of the boundary. If in addition $\beta = \pi$, i.e. the given wave of the form (1.3) moves along the boundary, we have $v \equiv u$, which means that the incident and reflected waves coincide, yielding the sum of the wave moving along the boundary. In this case we find that $A_0 = 1/2\, A\ (0, 0)$ in (4.4) where A is the first coefficient of the geometric acoustical expansion (1.3) of this wave.

During the derivation of (4.4) it was assumed that $\varphi > \pi - \beta$. Only for these values of φ the wave w_0 and the correction w coincide (see Sect. 1). The formula (4.4) however is also valid for $\varphi < \pi - \beta$. To prove this, the line $y = g_0(x) + \psi(x)$ must be taken as the analytic boundary Γ_0 and x replaced by $-x$. Then a, β, φ become $(-1)^{n+1} a$, $\pi - \beta$, $\pi - \varphi$, respectively, and the problem reduces to the case $\varphi > \pi - \beta$, for which (4.4) has been already proved. The substitution just shown does not alter the formula, and consequently the latter is valid for $\varphi < \pi - \beta$.

5. Let us now consider the problem formulated in Sect. 1, with the boundary condition $U = 0$ on Γ. In this case the function w satisfying the condition $w = -q(t, x)$ on Γ, is expressed by

$$w(T, X, Y) = -\iint \frac{\partial}{\partial t} q(t, x) \frac{\partial}{\partial n} G^*(T - t, X, Y; x, y)\, dt ds \quad (5.1)$$

The region of integration is defined as in (1.5) and G^* denotes the Green's function for the wave equation with the boundary condition $U = 0$, integrated once with respect to T. The formula (5.1) can be obtained from formula (66) Sect. 502 of [2], by replacing v by the function $2\pi G^*$ in the latter and can be proved in a similar fashion since both functions have the same singularity on the axis of the characteristic cone.

The integration in (5.1) is performed as in (1.5) and in place of (4.4) we obtain

$$w_0 = -\frac{\sqrt{2}\, a A_0 h^{n+m-1/2} \sin\beta \sin\varphi}{\sqrt{\pi r}\, \Gamma(n + m + 1/2)(-\cos\beta - \cos\varphi)^{n+1}} + O(h^{n+m+1/2}) \quad (5.2)$$

6. Certain modifications of the problem will now be considered. It is assumed, for definiteness, that the boundary condition $\partial U / \partial n = 0$ applies. Let us assume that in (1.2) $n > 1$ is not an integer and replace $n!$ by $\Gamma(n + 1)$, so that the function $\psi(x)$ is analytic only when $x > 0$.

In this case the formula (4.4) remains valid in the region $\varphi > \pi - \beta$, the estimate of the remainder deteriorating only for $1 < n < 2$. In the region $\varphi < \pi - \beta$ the integral in (1.5) yields the sum of the incident and reflected wave. The region of integration remains finite when $h \to 0$ and the nonanalytic part of the integral must be isolated to find the singularity of the diffracted wavefront. In general it is found that the singularity in the region $\varphi < \pi - \beta$ appears on both sides of the wavefront $T = r$, i.e. as $r \to T - 0$ and as $r \to T + 0$.

Assume now that $\psi(x)$ decreases faster than any power of x as $x \to 0$ and ψ'' is monotonic at small x. Then, instead of (4.4), we obtain for $\varphi > \pi - \beta$

$$w_0 = \frac{\sqrt{2}\, a A_0}{\sqrt{\pi r}} (1 + \cos\beta\cos\varphi)\, b^{m-3/2} \Psi(h/b) + O(h^2 \Psi''(h/b)) \quad (6.1)$$

where $\Psi(x) = I_{m-1/2}\psi(x)$, the operator I_k being given in (4.3), $\psi(x) = 0$ when $x \leqslant 0$; $h = T - r$ and $b = -\cos\beta - \cos\varphi > 0$.

Now consider the steady state problem with the boundary condition $\partial U / \partial n = 0$. Assume that the geometric acoustical expansion of the incident wave has the form $u = A(x, y)\, e^{-i\omega(t - p(x,y))} + \ldots$, i.e. that it can be obtained from (1.3) by replacing $f_0(\tau)$ by $e^{-i\omega\tau}$. Performing the corresponding formal substitution in (4.4), we obtain

$$w_0(T, X, Y) = e^{-i\omega T} \frac{\sqrt{2}\, a A_0 e^{i(\omega r + n\pi/2 - \pi/4)} (1 + \cos\beta\cos\varphi)}{\sqrt{\pi r}\, \omega^{n-1/2} (-\cos\beta - \cos\varphi)^{n+1}} + O(\omega^{-n-1/2}) \quad (6.2)$$

In the case when $x < 0$, the boundary is a straight line, becoming a curve when $x \geqslant 0$ and $\beta = \pi$; this problem has been solved using approximate methods in [3, 4]

It is also noteworthy that the known Kirchhoff method yields an expression for w_r, which agrees with (6.2) with the accuracy of up to $O(\omega^{-n-1/2})$.

7. A rigorous estimate of the change in the value of the integral (1.5) caused by neglecting G_1 may now be obtained for the case when $m \geqslant 6$ in (2.1). Integrating (1.5) by parts four times we obtain

$$w(T, X, Y) = \iint \frac{\partial^4 q(t,x)}{\partial t^4} H(T - t, X, Y; x, y)\, dt\, ds \qquad (7.1)$$

By (3.1), $H = H_0 + H_1$; functions H and H_i are obtained from G and G_i by fourfold integration with respect to T. Since the Green's function G satisfies the boundary condition $\partial G / \partial n = 0$, it follows that H also obeys this condition. Consequently on Γ we have

$$\partial H_1 / \partial n = -q^* (t, x) \qquad (q^* \equiv \partial H_0 / \partial n|_\Gamma) \qquad (7.2)$$

Now replacing $T - t$ by t and considering $H_1 (t, X, Y; x, y)$ as a function of t, x, y we find that, as in Sect. 3, $H_0 = H_2 + H_3$, where

$$H_2 = (2\pi\rho)^{-1/2} (t - \rho)^{m_1} / \Gamma (m_1 + 1) + O ((t - \rho)^{m_1+1}), \qquad m_1 = 7/2 \qquad (7.3)$$

and H_3 is the wave formed on reflection of H_2 from Γ_0. Formula (2.6) can therefore be used with

$$m = 7/2, \qquad A_0 = (2\pi r)^{-1/2}, \qquad \beta = \varphi, \qquad r = (X^2 + Y^2)^{1/2}$$

to estimate the function q^*.

If the function q^* and its first derivatives are now inspected in the region $|t - r| + |x| < Ch$ where h are small, we find that since by (2.9) $\psi' = O(h^{n-1})$, $\tau = O(h)$, $\tau - \tau_0 = O(h^n)$ in this region, Eq. (2.6) yields $q^* = O(h^{n+3/2})$. The derivatives

$$q_t^* = O(h^{n+1/2}), \qquad q_x^* = O(h^{n+1/2}) \qquad (7.4)$$

are estimated in a similar fashion.

Let us now consider the function H_1. This function satisfies the wave equation, the boundary condition (7.2) and vanishes when $H = H_0$, i.e. in the regions not yet reached by the waves formed on reflection of (7.3) from the segments of the boundaries Γ and Γ_0 on which $x > 0$. If H_1 is written in the form $H_1 = V + W$, where V is a function constructed below satisfying the boundary condition (7.2) and W satisfies the homogeneous boundary condition as well as the following nonhomogeneous equation

$$W_{tt} - \Delta W = f \quad (f \equiv \Delta V - V_{tt}), \qquad \partial W / \partial n = 0 \quad \text{on } \Gamma \qquad (7.5)$$

Let $\omega(\tau, \xi)$ denote any function of class C^2 positive in the region Q ($1 < \tau < 2$, $0 < \xi < 1$), vanishing outside Q and such that its integral over Q is equal to unity. In addition, let n denote a normal to Γ,

$$\eta = y - g(x), \qquad q^\circ (t, x) = q^* (t, x) \cos (n, y)$$

$$V (t, x, y) = -\eta \iint q^\circ (t - \eta\tau, x - \eta\xi)\, \omega (\tau, \xi)\, d\tau\, d\xi$$

It is now possible to estimate the derivatives of V

$$V_y = V_\eta = -\iint (q^\circ - \eta\tau q_t^\circ - \eta\xi q_x^\circ)\, \omega d\tau d\xi$$

On Γ, i.e. when $\eta = 0$, we have $V = 0$, $V_y = -q^\circ$, therefore $\partial V / \partial n = -q^*$ on Γ. Furthermore $q_\tau^\circ = -\eta q_t^\circ$, $q_\xi^\circ = -\eta q_x^\circ$, therefore on integrating by parts we obtain

$$V_y = \iint q^\circ \; (\omega + \tau\omega_\tau + \xi\omega_\xi)\, d\tau d\xi$$

V_t and V_x can be similarly expressed in terms of q^* and the second derivatives of V in terms of the first derivatives of q°. It follows therefore from (7.4) and (7.5) that $f = O(h^{n+1/2})$.

To estimate the energy of the wave W we denote by K the part of the cone

$$K_0 \left(-C_2 h < t - r < C_1 h - \sqrt{x^2 + y^2}\right)$$

in which $y > g(x)$; by $K(t)$ its intersection with the plane $t = \text{const}$ and assume that r and φ are as in (3.4). The number C_2 is chosen so that $H_1 = V = 0$ on the cone base, i.e. on $K(t_0)$ where $t_0 = r - C_2 h$.

Such a choice is feasible when $0 < h < h_0$, since $H_1 = H - H_0 = 0$ outside the region of influence of those segments of Γ and Γ_0 where $x > 0$. On the segments themselves we have that $H = H_0 = 0$ in the parts not yet reached by the wave (7.3), i.e. where $t - r < -x \cos\varphi + O(x^2)$. The number C_2 can be taken as independent of φ and h for any fixed $\delta > 0$ when $\delta \leqslant \varphi \leqslant \pi$.

Let $E(t)$ denote the energy of the wave W in the region $K(t)$, i.e.

$$E(t) = \frac{1}{2} \iint\limits_{K(t)} (W_t^2 + W_x^2 + W_y^2)\, dx\, dy \tag{7.6}$$

Since $W = W_t = 0$ in $K(t_0)$ we have, as in [5],

$$E(t) \leqslant \left[\int\limits_{t_0}^{t} \left(\frac{1}{2} \iint\limits_{K(t_1)} f^2 dx\, dy \right)^{1/2} dt_1 \right]^2 \leqslant \frac{t - t_0}{2} \iiint\limits_{K} f^2\, dx\, dy\, dt.$$

Since $t - t_0 < C_1 h + C_2 h$, $f = O(h^{n+1/2})$, then $E(t) = O(h^{2n+5})$.

Let us estimate the function W on Γ_h, the part of Γ contained inside the prism $|t - r| + |x| < C_0 h$. For a sufficiently large ratio $C_1 : C_0$, $W = 0$ on the part S_h of the side surface of the cone K_0 lying inside the prism. Therefore on Γ_h

$$H_1^2 = W^2 = \left(\int W_y dy\right)^2 \leqslant (C_1 + C_2)\, h \int W_y^2 dy$$

Here the integration is performed over the segment connecting Γ_h and S_h. Integrating both sides of the inequality with respect to t and x over the region Γ_h and noting that by virtue of (7.6) the triple integral of W_y^2 is not larger than the integral of $2E(t)$, we obtain

$$\iint\limits_{\Gamma_h} H_1^2\, dt\, dx = O(h^{2n+7}) \tag{7.7}$$

Now estimate the part J_1 of the integral (7.1), obtained by replacing H by H_1, which was neglected in the derivation of (4.4). Since $q = \partial(u + v) / \partial n|_\Gamma$, it follows from (2.1) and (2.6) that $\partial^4 q / \partial t^4 = O(h^{n+m-6})$. According to formulas (3.6)–(3.8), when $\varphi > \pi - \beta + \delta$ the integration in (7.1) is performed over the region with radius of the order of $O(h)$ surrounding the coordinate origin. Using the Buniakowski inequality and the estimates obtained, we find that $J_1 = O(h^{2n+m-3/2})$. Since $n \geqslant 2$, J_1 can be included in the remainder of formula (4.4).

From the estimate obtained it also follows that when $n > 2$, we can find another $n - 2$ terms of the geometric acoustical expansion for the wave (4.4) by replacing G in (1.5) by the known function G_0.

8. For the boundary condition $U = 0$ a rigorous estimate of the influence of the neglected terms is obtained in basically the same manner. Let $m \geqslant 4$ in (2.1). Double integration by parts of (5.1) yields

$$w(T, X, Y) = -\iint \frac{\partial^3 q(t, x)}{\partial t^3} \frac{\partial}{\partial n} H(T - t, X, Y; x, y)\, dt\, ds \tag{8.1}$$

As in (3.1) we find that $H = H_0 + H_1$; the functions H and H_i are obtained from G and G_i by integrating three times with respect to T. Relation (7.2) on Γ now becomes $H_1 = -q^*(t, x)$, $q^* = H_0$, and $m_1 = 5/2$ in (7.3). Estimates (7.4) remain as before but the estimate for H_1 changes.

Lemma. Let K denote the part of the cone $t_0 \leqslant t \leqslant t_1 - \sqrt{x^2 + y^2}$ in which $y > g(x)$ and $|g'(x)| \leqslant$ const and B is the part of the surface $y = g(x)$ situated inside the cone. Also let u be the solution of $u_{tt} = u_{xx} + u_{yy}$ in K satisfying the conditions $u = u_t = 0$ when $t = t_0$ and $u = \varphi$ on B. Then

$$\iint\limits_B \left(\frac{\partial u}{\partial n}\right)^2 \alpha\, ds\, dt \leqslant 2 \iint\limits_B \frac{(1+|\beta|)^2}{\alpha} \left[\left(\frac{\partial \varphi}{\partial t}\right)^2 + \left(\frac{\partial \varphi}{\partial s}\right)^2\right] ds\, dt \tag{8.2}$$

$$\alpha = \cos(n, y) = (1 + g'^2)^{-1/2}, \quad \beta = \cos(n, x) = -\alpha g'$$

where s is the arc length of the curve $y = g(x)$.

The lemma is proved by integrating the identity $2(u_t + u_y)\ (u_{tt} - u_{xx} - u_{yy}) \equiv 0$ over K and transforming the resulting volume integrals into surface integrals.

Note. Using the inequality (8.2) we can easily estimate the energy $E(t)$ of the wave u at any cross section $t = t_2$ of K

$$E(t_2) \equiv \frac{1}{2} \iint\limits_{K(t_2)} (u_t^2 + u_x^2 + u_y^2)\, dx\, dy = - \iint\limits_{B\,(t < t_2)} \frac{\partial u}{\partial n} \frac{\partial \varphi}{\partial t}\, dt\, ds \tag{8.3}$$

The estimate for the right-hand side is obtained using (8.2) and the Buniakowski inequality.

Applying the lemma to the case when $u = H_1$, $\varphi = q^*$ and K is the same as in Sect. 7, we obtain

$$\iint\limits_{\Gamma_h} \left(\frac{\partial H_1}{\partial n}\right)^2 ds\, dt \leqslant C \iint\limits_{\Gamma_h} \left[\left(\frac{\partial q^*}{\partial t}\right)^2 + \left(\frac{\partial q^*}{\partial x}\right)^2\right] ds\, dt = O(h^{2n+3})$$

Taking into account the fact that in (2.1) $m \geqslant 4$, we obtain $\partial^3 q / \partial t^3 = O(h^{n+m-4})$ on Γ_h. Consequently the integral of $\partial^3 q / \partial t^3 \partial H_1 / \partial n$ over Γ_h is equal to $O(h^{2n+m-5/2})$ and can be included in the remainder of (5.2) when $n \geqslant 2$.

9. The method given above can also be used when the boundary Γ consists of two smooth arcs meeting at an angle at the point O, i.e. the method is also applicable to the problem of the diffraction of a wave in an angular region with curved sides. In this case, the angle formed by two tangents to the curves at O is taken as Γ_0, and the first term of the geometric acoustical expansion of the diffracted wave is identical with that obtained in the problem of diffraction on Γ_0. The proposed method makes it possible to obtain the second term and a rigorous estimate of the remainder (at that portion of the diffracted wavefront at which the reflected wave is absent).

Suppose that two terms of the geometric acoustical expansion of the incident wave (1.3) are known. The solution of the problem of diffraction of this wave on Γ is the sum of the known solution (see [5] Sect. 4) of the diffraction problem of this wave on Γ_0 and the correction w defined by (1.5) or (5.1). As before we have that $G = G_0 + G_1$ where G_0 is a known [5] Green's function for the region bounded by Γ_0 and G_1 is the remainder. As in Sects. 4 and 5 we have $w = O(h^{n+m-1/2})$, and neglecting G_1 causes an error in w of the order of $O(h^{2n+m-3/2})$ (the proof is identical to that of Sects. 7 and 8), and does not affect the second term of the geometric acoustical expansion of the diffracted wave, this term being of the order of $O(h^{m+3/2})$.

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ON A FORM OF STEADY CAPILLARY-GRAVITATIONAL WAVES OF FINITE AMPLITUDE

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A problem concerning steady, capillary-gravitational waves of finite amplitude generated by pressure periodically distributed over the surface of an infinitely deep stream is considered. A rigorous solution of this problem is presented, with the surface pressure given, in the form of an infinite trigonometric series. In addition a particular case is investigated when the wavelength of the given pressure coincides with the length of the steady free wave corresponding to the specified flow velocity and constant pressure at the surface. The waves investigated here cease to exist when the periodic part of the pressure distributed over the surface vanishes identically and the flow becomes uniform. Such waves have been called induced [1]. An analogous problem for gravitational waves was investigated earlier [2] by the suthor. In addition, the author used the Levi-Civita method [3, 4] to reduce a similar problem for free capillary-gravitational waves, to a nonlinear differential equation.

In the present paper the problem is reduced to solving a certain nonlinear integral equation. The latter is discussed and its solution is constructed for any degree of approximation. The first three approximations are derived completely and an approximate equation describing the wave profile is given.

1. Statement of the problem and derivation of the basic integral equation. Consider a plane parallel steady motion of a perfect incompressible heavy fluid bounded only from above by a free surface at which the pressure is given by $p_0 = p_0' + p_0(x)$. Here $p_0' = \text{const}$ and $p_0(x)$ is a given periodic function of the horizontal coordinate x. The flow is assumed to move from left to right with constant velocity c, at an infinite depth. Since the pressure at the surface is a periodic function of x, the surface assumes the form of a periodic wave, stationary with respect to coordinates attached to a progressive wave moving with velocity c. The present paper shows that induced waves exist for any finite values of c.

Let the required wave and the pressure $p_0(x)$ both possess the same symmetry with respect to the vertical through the wave crest. The y-axis is chosen so as to coincide with the axis of symmetry, and is directed vertically upwards. The coordinate origin O is placed at the point of intersection of y-axis with the free surface and the x-axis is directed to the right.